# SURJECTIVE ISOMETRIES ON A BANACH SPACE OF ANALYTIC FUNCTIONS ON THE OPEN UNIT DISC (Researches on isometries as preserver problems and related topics)



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# SURJECTIVE ISOMETRIES ON A BANACH SPACE OF ANALYTIC FUNCTIONS ON THE OPEN UNIT DISC

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This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

## 1. Introduction

Let $(M, \|\cdot\|_M)$ and $(N, \|\cdot\|_N)$ be normed linear spaces, respectively. A mapping $T\colon (M, \|\cdot\|_M) \to (N, \|\cdot\|_N)$ is an isometry if and only if it preserves the distance of two points in $M$, that is,

$$\|T(a) - T(b)\|_N = \|a - b\|_M \qquad (a, b \in M).$$

Here, we assume that $T$ is not necessarily complex linear. The Mazur-Ulam theorem [16] states that every surjective isometry $T$ between two normed linear spaces is real linear provided $T(0) = 0$.

We mention the characterization of isometries on several normed linear spaces. Isometries were studied on various spaces by many researchers, as for example in [3, 12, 13, 21, 22]. In 1932, isometries are studied by Banach [1, Theorem 3 in Chapter XI] (see also [24, Theorem 83]). There have been numerous papers on isometries defined on Banach spaces of analytic functions; see [2, 4, 5, 8, 11, 14].

Among the basic problems in analytic function spaces, Novinger and Oberlin, in [20], characterized *complex linear* isometries on a normed space $\mathcal{S}^p$. The underlying space $\mathcal{S}^p$ is a normed space consisting of analytic functions $f$ on the open unit disc $\mathbb{D}$ whose derivative $f'$ belongs to the classical Hardy space $(H^p(\mathbb{D}), \|\cdot\|_p)$ for $1 \leq p < \infty$. They introduced the norm $|f(0)| + \|f'\|_p$ on the normed space $\mathcal{S}^p$.

In this talk, we study surjective isometries on the Banach space $\mathcal{S}_A$ of analytic functions $f$ defined on $\mathbb{D}$ whose derivative can be extended to the closed unit disc $\bar{\mathbb{D}}$, and endowed with the norm $\|f\|_\sigma = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|$. We denote by $A(\bar{\mathbb{D}})$ the disc algebra, that is, the algebra of all analytic functions on $\mathbb{D}$ which can be extended to continuous functions on $\bar{\mathbb{D}}$.

## 2. Main result

Let $A(\bar{\mathbb{D}})$ be the Banach space of all analytic functions on the open unit disc $\mathbb{D}$ that can be continuously extended to the closed unit disc $\bar{\mathbb{D}}$ with the supremum norm on $\mathbb{D}$. For each $v \in A(\bar{\mathbb{D}})$, $v'$ means the derivative of $v$ on $\mathbb{D}$, that is,

$$v'(z) = \lim_{h \to 0} \frac{v(z+h) - v(z)}{h} \qquad (z \in \mathbb{D}).$$

We define $\mathcal{S}_A$ by the linear space of all analytic functions $f$ on $\mathbb{D}$ whose derivative $f'$ belongs to $A(\bar{\mathbb{D}})$. By [6, Theorem 3.11], we see that $\mathcal{S}_A \subset A(\bar{\mathbb{D}})$. By the definition of $\mathcal{S}_A$, $f'$ is an analytic function on $\mathbb{D}$ which can be extended to a continuous function on $\bar{\mathbb{D}}$. Let $\widehat{v}$ be the unique continuous extension of $v \in A(\bar{\mathbb{D}})$ to $\bar{\mathbb{D}}$. In fact, such an extension is unique since $\mathbb{D}$ is dense in $\bar{\mathbb{D}}$. We define the norm $\|f\|_\sigma$ of $f \in \mathcal{S}_A$ by

$$\|f\|_\sigma = |f(0)| + \|\widehat{f'}\|_\infty \qquad (f \in \mathcal{S}_A), \tag{2.1}$$

where $\|\widehat{f'}\|_\infty = \sup\{|\widehat{f'}(z)| : z \in \bar{\mathbb{D}}\} = \sup\{|f'(z)| : z \in \mathbb{D}\}$. It is not difficult to check that $(\mathcal{S}_A, \|\cdot\|_\sigma)$ is a complex Banach space.

**Theorem 1.** *If $T\colon (\mathcal{S}_A, \|\cdot\|_\sigma) \to (\mathcal{S}_A, \|\cdot\|_\sigma)$ is a surjective, not necessarily complex linear, isometry, then one of the following four forms is occured;*

*there exist constants $c_{1,1}, c_{1,2}, \lambda_1 \in \mathbb{T}$ and $a_1 \in \mathbb{D}$ such that*

$$T(f)(z) = T(0)(z) + c_{1,1} f(0) + \int_{[0,z]} c_{1,2} f'(\rho(\zeta))\, d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

*there exist constants $c_{2,1}, c_{2,2}, \lambda_2 \in \mathbb{T}$ and $a_2 \in \mathbb{D}$ such that*

$$T(f)(z) = T(0)(z) + \overline{c_{2,1} f(0)} + \int_{[0,z]} c_{2,2} f'(\rho(\zeta))\, d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

*there exist constants $c_{3,1}, c_{3,2}, \lambda_3 \in \mathbb{T}$ and $a_3 \in \mathbb{D}$ such that*

$$T(f)(z) = T(0)(z) + c_{3,1} f(0) + \int_{[0,z]} \overline{c_{3,2} f'(\rho(\overline{\zeta}))}\, d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

*there exist constants $c_{4,1}, c_{4,2}, \lambda_4 \in \mathbb{T}$ and $a_4 \in \mathbb{D}$ such that*

$$T(f)(z) = T(0)(z) + \overline{c_{4,1} f(0)} + \int_{[0,z]} \overline{c_{4,2} f'(\rho(\overline{\zeta}))}\, d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

*where $\rho(z) = \lambda_j \dfrac{z - a_j}{\overline{a_j} z - 1}$ for all $z \in \bar{\mathbb{D}}$ and for $j = 1, 2, 3, 4$.*

*Conversely, each of the above forms is a surjective isometry on $\mathcal{S}_A$ with the norm $\|\cdot\|_\sigma$, where $T(0)$ is an arbitrary element of $\mathcal{S}_A$.*

We start by defining an embedding of $\mathcal{S}_A$ into a subspace $B$ consisting of complex valued continuous functions. Then using the Arens-Kelley theorem (see [10, Corollary 2.3.6 and Theorem 2.3.8]), we give a characterization of extreme points of the unit ball $B_1^*$ of the dual space $B^*$ of $B$. Then we construct some maps to describe extreme points of $B_1^*$.

We used an idea by Ellis for the characterization of surjective real linear isometries on uniform algebras (see [9]). An adjoint operator of a surjective real linear isometry on the dual space $B^*$ preserves extreme points. The action of such adjoint operator on the set of extreme points gives a representation for the isometries on $B$. We show that the isometries of $\mathcal{S}_A$ are integral operators of weighted differential operators.

For the details of proof, refer to [18].

## References


[1] S. Banach, Theory of linear operations, Translated by F. Jellett, Dover Publications, Inc. Mineola, New York, 2009.

[2] F. Botelho, *Isometries and Hermitian operators on Zygmund spaces*, Canad. Math. Bull. **58** (2015), 241–249.

[3] M. Cambern, *Isometries of certain Banach algebras*, Studia Math. **25** (1964-1965) 217–225.

[4] J.A. Cima and W.R. Wogen, *On isometries of the Bloch space*, Illinois J. Math. **24** (1980), 313–316.

[5] K. deLeeuw, W. Rudin and J. Wermer, *The isometries of some function spaces*, Proc. Amer. Math. Soc. **11** (1960), 694–698.

[6] P.L. Duren, The theory of $H^p$ spaces, Academic Press, New York, 1970

[7] F. Forelli, *The isometries of $H^p$*, Canad. J. Math. **16** (1964), 721–728.

[8] F. Forelli, *A theorem on isometreis and the application of it to the isometries of $H^p(S)$ for $2 < p < \infty$*, Canad. J. Math. **25** (1973), 284–289

[9] A.J. Ellis, *Real characterizations of function algebras amongst function spaces*, Bull. London Math. Soc. **22** (1990), 381–385.

[10] R. Fleming and J. Jamison, *Isometries on Banach spaces: function spaces*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 129. Chapman & Hall/CRC, Boca Raton, FL, 2003.

[11] W. Hornor and J.E. Jamison, *Isometreis of some Banach spaces of analytic functions*, Integral Equations Operator Theory **41** (2001), 410–425.

[12] K. Jarosz and V.D. Pathak, *Isometries between function spaces*, Trans. Amer. Math. Soc. **305** (1988), 193–205.

[13] K. Kawamura, H. Koshimizu and T. Miura, *Norms on $C^1([0,1])$ and their isometries*, Acta Sci. Math. (Szeged) **84** (2018), 239–261.

[14] C.J. Kolaski, *Isometries of Bergman spaces over bounded Runge domains*, Canad. J. Math. **33** (1981), 1157–1164.

[15] H. Koshimizu, *Linear isometries on spaces of continuously differentiable and Lipschitz continuous functions*, Nihonkai Math. J. **22** (2011), 73–90.

[16] S. Mazur and S. Ulam, *Sur les transformationes isométriques d'espaces vectoriels normés*, C. R. Acad. Sci. Paris **194** (1932), 946–948.

[17] T. Miura, *Surjective isometries between function spaces*, Contemp. Math. **645** (2015), 231–239.

[18] T. Miura and N. Niwa, *Surjective isometries on a Banach space of analytic functions on the open unit disc*, Nihonkai Math. J. **29** (2018), 53–67.

[19] M. Nagasawa, *Isomorphisms between commutative Banach algebras with an application to rings of analytic functions*, Kōdai Math. Sem. Rep. **11** (1959), 182–188.

[20] W.P. Novinger and D.M. Oberlin, *Linear isometries of some normed spaces of analytic functions*, Canad. J. Math. **37** (1985), 62–74.

[21] V.D. Pathak, *Isometries of* $C^{(n)}[0,1]$, Pacific J. Math. **94** (1981), 211–222.

[22] N.V. Rao and A.K. Roy, *Linear isometries of some function spaces*, Pacific J. Math. **38** (1971), 177–192.

[23] W. Rudin, Real and complex analysis. Third edition. McGraw-Hill Book Co., New York, 1987.

[24] M.H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. **41** (1937), 375–481.